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Self-similarity and fractals in soliton-supporting systems

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We describe a principle that can be used to generate self-similarity and fractals in almost any nonlinear system in nature that supports solitons, given that some proper nonadiabatic conditions are met. We illustrate our idea on a particular optics example that also theoretically demonstrates fractals in nonlinear optics.

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Fractals are one of the most fundamental concepts in nature [1], characterizing many natural phenomena; they have been described not only in biology, medicine, galactic clusters, material structures, etc., but also in areas as surprising as stock markets [2]. In optics, fractals have been identified in conjunction with binary gratings [3] and with unstable cavity modes [4]. Both of these optical fractal systems are fully linear; they respond in a passive manner to illumination by constructing fractals through linear diffraction. In this Rapid Communication, we show that nonlinear systems that support solitons can, under proper nonadiabatic conditions, *evolve* and give rise to statistical fractals. Further, our idea can be used to demonstrate exact fractals as well. The principle we describe is universal and seems to hold for most soliton-supporting systems in nature. Just as an illustration of our idea, we present a specific example that theoretically demonstrates fractals in nonlinear optics.

As an example of a soliton-supporting system, consider a system described by the normalized nonlinear Schrödinger equation (NLSE):

$$i\frac{\partial \Psi}{\partial z}+\frac{1}{2}\nabla_T^2\Psi+f(|\Psi|^2)\Psi=0, \qquad (1)$$

where the nonlinear term $f(|\Psi|^2)$ is specific to the physical system, and ∇_T^2 is the Laplacian transverse to the propagation direction z. For waves in a single transverse dimension [(1+1)D *NLSE*], $\nabla_T^2=\partial^2/\partial x^2$. Equation (1) describes many physical systems, primarily those in which nonlinear waves propagate in isotropic media [5], where Ψ describes the slowly varying envelope that modulates a fast carrier wave. In particular, NLSE describes several optical systems [6,7], where Ψ is the slowly varying amplitude of the electric field, superimposed on a single **k** vector carrier plane wave. We focus on the (1+1)D NLSE and on two particular forms of nonlinearity that are common in optics [7]: the Kerr-type, where $f(|\Psi|^2)=|\Psi|^2$, and the saturable type, where $f(|\Psi|^2)=|\Psi|^2/(1+|\Psi|^2)$. Extending our ideas to other forms of nonlinearities is straightforward, and extending them to higher dimensions maintains the main results while adding beauty and complexity to the fractals generated. The particular systems we discuss are just examples of the general principle we propose.

Since $df(|\Psi|^2)/d|\Psi|^2>0$, both the Kerr and the saturable nonlinearity are of the self-focusing type, i.e., the nonlinearity has a tendency to shrink a pulse. In optical systems, this happens because the presence of the light pulse increases the local index of refraction, which, in turn, tends to shrink the pulse. The tendency to shrink competes with diffraction, which tries to expand the pulse, and, for some NLSEs, these two tendencies can exactly cancel each other, producing a localized pulse whose shape is stationary as it propagates: a soliton [8]. Solitons are universal nonlinear phenomena, and they have fascinated scientists of different fields for more than 150 years now [9]. They have been described in many systems: on the surface of shallow water [9], in deep sea water [5], in plasma [10], and on the surface of black holes [11], to name a few, and of course in nonlinear optics, primarily as temporal [6] and as spatial solitons [7]. The solitons of the particular NLSEs we discuss in this article are very robust creatures. Even if one perturbs them slightly from their equilibrium shape, they soon evolve into stable solitons again.

Consider first the (1+1)D Kerr NLSE, of which a fundamental soliton solution is $\Psi(x,z)$. One can obtain a whole family of solitons of Eq. (1) by a simple rescaling: $\Psi(x,z)\rightarrow q\Psi(qx,q^2z)$ for any real q. According to the definition of self-similarity, this means that all solitons of the same order of this equation are self-similar to each other [12]. This property, in fact, holds for solitons of any order N: if $\Psi(x,z=0)$ is a soliton of order 1, then $N\Psi(x,z=0)$ is a soliton of order N. Furthermore, because the generic waveform of solitons of all orders is hyperbolic secant, then solitons of different orders are also self-similar to one another, at least at some points in their propagation (although their propagation dynamics differs from one order to another). The physical basis for this self-similarity is the fact that the (1+1)D Kerr NLSE does not have any natural scale built into it, so the physics of this equation looks the same on all scales.

In contrast to the Kerr nonlinearity, the (1+1)D saturable NLSE does have a natural scale, given by the number 1 in the denominator of the nonlinear term. However, for $|\Psi|^2\ll 1$, the nonlinearity reduces to the Kerr nonlinearity, so

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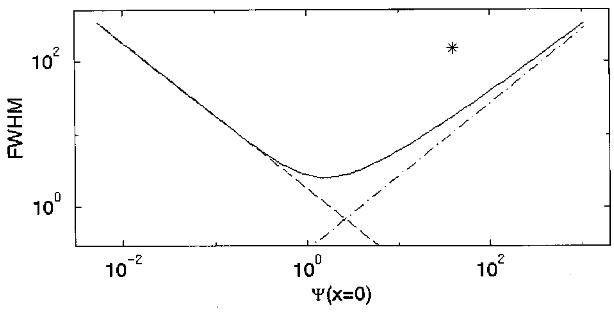

FIG. 1. Existence curves of Kerr-type solitons (dashed line), solitons in a saturable nonlinear medium (solid curve), and deep-saturation solitons (dashed-dotted line), all in (1+1)D. The vertical axis gives the normalized width of the intensity of the soliton, in the x-units of Eq. (1). The point indicated by $*$ describes the input pulse to the fractal-generating process of Fig. 2.

self-similarity exists in the saturable case also. Furthermore, if a soliton of (1+1)D saturable NLSE satisfies $|\Psi(x=0,z)|^2 \gg 1$ in the regions where most of the energy of the soliton is contained, $|\Psi|^2/(1+|\Psi|^2) \approx 1-(1/|\Psi|^2)$. Of course, this approximation does not hold at the tails of the soliton. Still, most of the interesting properties can be captured by studying (1+1)D NLSE with a nonlinearity given by $1-(1/|\Psi|^2)$; we call this nonlinearity the ''deep-saturation nonlinearity.'' We have checked numerically that if the condition $|\Psi(x=0,z)|^2 \gg 1$ is satisfied, then indeed most of the soliton's physics is captured by studying the (1+1)D deep-saturation NLSE. If $\Psi(x,z)$ is a solution of the (1+1)D deep-saturation NLSE, then a whole family of solutions can be obtained by re-scaling $\Psi(x,z) \rightarrow e^{iz(1-q^2)}\Psi(qx,q^2z)/q$, for any real q. Therefore, all solitons of the same order of (1+1)D saturable NLSE are related by this simple rescaling, as long as most of the energy of the solitons is in the regions where $|\Psi|^2 \gg 1$; all of these solitons are self-similar to one another in their physical properties, such as intensity, shape, etc. This is because the natural scale in the saturable NLSE is visible only in the margins of the intensity profile of the soliton, and its effect on the shape is tiny.

We now introduce the concept of the soliton existence curve [13], a two-dimensional curve that gives the full width at half-maximum of the soliton intensity in normalized units, as a function of the peak amplitude of the corresponding soliton, $\Psi_0 \equiv \Psi(x=0,z)$. The curve is drawn for the set of all solitons of the same order of a given NLSE, where each soliton is represented by a point on the graph. Different NLSEs have different existence curves, and solitons of different orders of the same NLSE lie on different existence curves. According to the scaling relation described above, all existence curves (of solitons of all orders) of Kerr NLSEs are parallel lines of slope -1 on a log-log plot. The existence curves of saturable NLSEs are also parallel lines of slope -1 on a log-log plot (which coincide with the Kerr curves) in the region $\Psi_0 \ll 1$. On the other hand, in deep-saturation where $\Psi_0 \gg 1$, the existence curves are parallel lines of slope 1 on a log-log plot, like in Fig. 1. The region in between these two regimes, i.e., where $\Psi_0 \sim 1$, we call the valley. All solitons of the same order of a saturable NLSE are to a large extent self-similar to each other as long as they are all on the same side of the valley.

The existence curves can provide information about the evolution of arbitrary input pulses into solitons. Consider a pulse of width w and peak amplitude Ψ_0, and assume that this pulse does not have the stationary soliton shape. This pulse is represented by a point on the existence curve plot. If this point is close to the curve, then the pulse soon evolves into a stable soliton shape (while shedding some power in the form of radiation modes or smaller scale solitons). Since the solitons of the NLSEs we study here are stable, this happens even though their initial shape only approximates a soliton.

Having established that Kerr solitons are exactly self-similar, and that deep-saturation solitons are approximately self-similar, it is now compelling to ask: ''Can solitons of various scales coexist in the same nonlinear medium simultaneously?'' If the answer is positive, can they coexist within one another in a fractal structure? And, if the answer to this question is also positive, then how can a nonlinear system be driven to generate solitons organized in a fractal structure? The answers to all of these questions are in the response of the nonlinear system to a nonadiabatic change in one (or more) of its properties. As described below, an abrupt change in the nonlinear coefficient, or in the saturation coefficient (in saturable systems), or in the dispersion coefficient (for temporal solitons), or in almost any parameter that leads to a large deviation of the pulse from the soliton existence curve, will lead to the appearance of a fractal structure driven by soliton dynamics. In our simulations, we observe self-similarity and fractals both in the Kerr regime, and in the deep-saturation regime of Eq. (1). Here we present only the results in deep-saturation, which can be realized with the photorefractive nonlinearity (see Ref. [7], and references therein).

Our goal is to design a physical system that can support many solitons all of different sizes simultaneously. If one starts with a pulse whose shape is very far off the existence curve, this pulse is not able to evolve smoothly into a soliton. Under proper conditions, it breaks up into smaller pieces and radiation. Quite often, the pieces resulting from this ''explosion'' include many small solitons, all of different sizes. If the nonlinearity is such that these solitons are self-similar to each other, one can claim to have observed self-similarity.

We distinguish between two scenarios that produce such breakup. The first is driven by noise and is easier to realize experimentally. Consider a pulse whose initial width is far above the existence curve launched into a nonlinear medium in the regime that can support self-similar solitons (as in Fig. 1). Therefore, small perturbations (initiated by noise) of large wavelengths will grow on top of the pulse as it propagates. After some distance, the energy in these perturbations becomes significant, and the pulse breaks up into smaller pulses. This phenomena is known [6] as modulational instability (MI). In many physical systems, the products of this breakup include many solitons of different sizes. We call it ''MI-induced breakup.''

The second breakup scenario is a ''dynamics-induced breakup.'' It is observable in numerical calculations that inherently have no or very little noise. It should also be observable in ''clean'' experimental systems, such as temporal solitons in optical fibers. Consider a pulse above the exis-

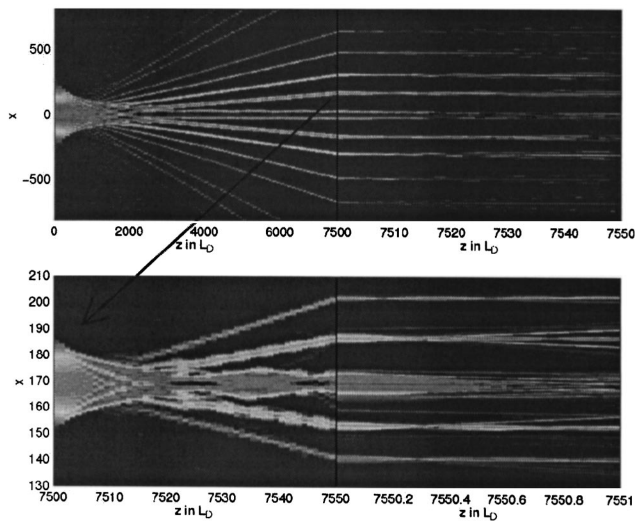

FIG. 2. Top view of a three-stage self-similar breakup creating a fractal structure *starting* in the deep-saturation regime of the (1 +1)D saturable NLSE. The 1st stage is given in the upper-left plot. The 2nd stage is given in the upper-right plot. A magnified detail of the 2nd stage is shown in the lower-left plot. The continuation of evolution of that detail into the 3rd stage is given in the lower-right plot.

tence curve launched into a self-focusing medium with the initial pulse width much larger than the width of the lowest guided mode of this induced waveguide. The light coalesces towards the center trying to reach the solitonic shape, in which diffraction is exactly balanced by self-focusing. However, once the equilibrium is reached the pulse keeps shrinking because of its inertia. Since the equilibrium could not have been reached smoothly (the pulse is initially far above the existence curve), the pulse explodes into smaller pieces, which form smaller solitons of different sizes.

In our simulations, we observe both breakup mechanisms. As an example, we present the dynamics driven breakup shown in the upper-left plot of Fig. 2. Since both the underlying equation and the initial pulse obey left-right symmetry, the output multisoliton pulses also obey this symmetry (in contrast with a MI-induced breakup, since noise obeys no symmetry).

In order to create fractals, one can apply the logic that has caused this breakup in a repetitive manner. One can take the output ‘‘daughter solitons’’ at the end of the upper-left plot in Fig. 2, make an abrupt change in the nonlinear medium, and thus force each one of the daughter solitons to break up into a train of smaller solitons. This happens if the change moves the position of the daughter solitons far above the existence curve. Such a change in the nonlinear medium can be realized either by altering the intensity of the pulses abruptly, or by changing the coefficient in front of ∇_T^2 in Eq. (1), or by changing the properties of the nonlinearity (magnitude, saturation, etc.). It is important that the change in the conditions is abrupt; an adiabatic change does not cause a breakup, but instead the pulse adapts and evolves smoothly into a narrower soliton. When the change is abrupt and large enough, each of the pulses undergoes a self-similar breakup as illustrated in the top plot of Fig. 2. This process can be repeated, in principle, an infinite number of times, thereby creating a fractal structure. Of course, all the resulting solitons after each breakup have to be in the regime where they are all self-similar to each other.

A three stage fractal is presented in Fig. 2. This breakup is dynamics-induced. At the input, self-focusing is much stronger than diffraction for a pulse of that width, so the pulse contracts and eventually breaks up into many self-similar solitons observed at the output of the upper-left figure. At the plane of the output of the upper-left figure, we change the denominator in the nonlinear term from $1+|\psi|^2$ to $1+(|\psi|^2/8)$. This makes all the pulses at the output of the upper-left figure have amplitude eight times smaller than solitons of the same widths have. Then, we propagate the output of the upper-left figure for a few more diffraction lengths, resulting in a self-similar breakup of every pulse, as shown in the upper-right plot of Fig. 2. For better clarity, only a detail of the upper-right plot is shown in the lower-left plot. At the output of the upper-right figure, we change the nonlinear term into $1+(|\psi|^2/64)$, and propagate the pulse further. As shown in the lower-right figure, we observe one more stage of self-similar breakup. In these simulations, we use the saturable $|\psi|^2/1+|\psi|^2$ nonlinearity, to show when we expect the fractal generation process to end in a real system. In this case, the third stage shown by the lower-right figure is the final breakup, because most of the end solitons at this stage are of peak intensities on the order of unity, which ceases to be self-similar. As for the other end of this process, i.e., the first breakup, there is no upper limit: one can start this fractal generation process by literally breaking up plane waves.

To have a real fractal in mathematical sense, one should have a infinite number of stages in this process. However, as is the case with all other physical fractals, the number of stages is limited, thus resulting in a prefractal, rather than a fractal. The reason why all physical fractals live in only a limited regime of scales is easy to understand. Both at large, but primarily at small scales, sooner or later the scale of the fractal in question becomes comparable to some other relevant physical scale. Such scale then modifies the physics of the system, and the equation representing the system is modified, typically to an equation that does not display self-similarity anymore. For example, at small scales, at least the atomic scale presents a lower bound to fractal generation. Specifically for optical spatial solitons, the number of breakup stages is limited by the ratio between the beam width (of any of the daughters solitons in a particular stage) and the optical wavelength in the medium. When this ratio is smaller than, say 5, the beam is no longer paraxial and one has to add other terms to the equation (and for ratio $\sim$1 the underlying equation becomes vectorial). For optical temporal solitons in fibers, the limiting factors are third order dispersion, and additional nonlinear processes (e.g., Raman scattering). In both cases, we realistically expect to observe 3–4 stages of breakup.

Let us go back now to the definition of a fractal [2] as ‘‘an object which appears self-similar under varying degrees of magnification. In effect, possessing symmetry across scale, with each small part replicating the structure of the whole.’’ It is obvious that the structures described in Fig. 2 are fractals: they are self-similar *within* each scale (i.e., the daughter solitons after each breakup), and they are also self-similar over at least three widely separated scales. Also, as we show by the increasing magnification (Fig. 2), each part breaks up again and again in a structure replicating the whole. It is obvious that we have found a way to generate fractals, which are driven by soliton dynamics.

Several topics merit discussion. We want to emphasize the resemblance of our fractals to Cantor sets [2]. This fractal structure (after several breakup processes) is actually a *randomized Cantor set*, because at any given stage self-similar structures (solitonlike pulses) of various scales coexist, with the distances between these pulses varying in what looks like a random manner, especially in the presence of significant noise. Furthermore, the process we described is pretty robust; in our simulations we did not have to be particularly careful about when to start each particular stage in order for our final creatures to be fractals. The presence of the noise does not harm the fractal generation either; in fact, as we explained above, one can use this idea to build random fractals that are in fact *noise-driven*. This brings in another point: the *fractal dimensions*. The fractal dimension can be estimated using box-counting method, and it is obvious that this dimension is (in general) not an integer. In any case, the fractal dimension is specific to the system and the initial conditions used. Because the purpose of the paper is to present a new *general* idea, we leave the issue of the fractal dimension to future work. Another topic has to do with *fractals in higher Euclidian dimensions*, e.g., in full 3D. Since *the ideas presented are not restricted by dimensionality*, they should be easy to translate into higher dimensions and we envision fractals that have two different transverse scales, that is, when the width of every pulse in x is much different than that in y. However, caution must be taken to make sure that the system can support (2+1)D soliton structures first. The saturable NLSE can support stable bright (2+1)D solitons [14], so we expect it to support 2D fractals as well. There are many other new ideas under research, but the main challenge is experimental: to demonstrate fractals.

In conclusion, we proposed a general scheme of generating fractals by the dynamics of solitons that undergo abrupt changes along their propagation paths [15]. This method of fractal generation is universal and should exist in any nonlinear system that can support solitons. Interestingly, this is one of the very few cases in which one can generate fractals experimentally and investigate them theoretically knowing all the physics involved. In fact, experiments are currently carried out in our laboratory to demonstrate fractals from spatial and temporal solitons.

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[15] This paper was initially submitted to Science magazine in January 1999. It was rejected based on the argument that this ''interesting paper should be published in a Physics journal, such as PRL. In the meantime, we have found a way to generate *exact (regular) fractals that form an exact Cantor set*. These results will appear in S. Sears, M. Soljacic, M. Segev, D. Krylow, and K. Bergman, Phys. Rev. Lett. (to be published).